Multiple sign-changing and semi-nodal solutions for coupled Schrödinger equations*

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Abstract

We study the following coupled Schrödinger equations which have appeared as several models from mathematical physics:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, \quad x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, \quad x \in \Omega, \\ u_1 = u_2 = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N (N = 2, 3)$ is a smooth bounded domain, λ_1, λ_2, μ_1, μ_2 are all positive constants. We show that, for each $k \in \mathbb{N}$ there exists $\beta_k > 0$ such that this system has at least k sign-changing solutions (i.e., both two components change sign) and k semi-nodal solutions (i.e., one component changes sign and the other one is positive) for each fixed $\beta \in (0, \beta_k)$.

1 Introduction

In this paper we study solitary wave solutions of the coupled Gross-Pitaevskii equations (cf. [7]):

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, \quad x \in \Omega, \ t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, \quad x \in \Omega, \ t > 0, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad j = 1, 2, \\ \Phi_j(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \ j = 1, 2, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N (N = 2, 3)$ is a smooth bounded domain, i is the imaginary unit, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant. System (1.1) arises in mathematical models from several physical phenomena, especially in nonlinear optics. Physically, the solution Φ_j denotes the j^{th} component of the beam in Kerr-like photorefractive media (cf. [1]). The positive constant μ_j is for self-focusing in the j^{th} component of the beam, and the coupling constant β is the interaction between the two components of the beam. Problem (1.1) also arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (cf. [13]). Physically, Φ_j are the corresponding condensate amplitudes, μ_j and β are the intraspecies and interspecies scattering lengths. Precisely, the sign of μ_j represents the self-interactions of the single state $|j\rangle$. If $\mu_j > 0$ as considered here, it is called the focusing case, in opposition to the defocusing case where $\mu_j < 0$. Besides, the sign of β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive, i.e., the interaction is attractive if $\beta > 0$, and the interaction is repulsive if $\beta < 0$.

To study solitary wave solutions of (1.1), we set $\Phi_j(x, t) = e^{i\lambda_j t} u_j(x)$ for $j = 1, 2$. Then system (1.1) is reduced to the following elliptic system

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \Omega, \\ u_1 = u_2 = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{1.2}$$

Definition 1.1. *We call a solution (u_1, u_2) nontrivial if $u_j \not\equiv 0$ for $j = 1, 2$, a solution (u_1, u_2) semi-trivial if (u_1, u_2) is type of $(u_1, 0)$ or $(0, u_2)$. A solution (u_1, u_2) is called positive if $u_j > 0$ in Ω for $j = 1, 2$, a solution (u_1, u_2) sign-changing if both u_1 and u_2 change sign, a solution (u_1, u_2) semi-nodal if one component is positive and the other one changes sign.*

In the last decades, system (1.2) has received great interest from many mathematicians. In particular, the existence and multiplicity of *positive* solutions of (1.2) have been well studied in [2, 3, 5, 6, 9, 12, 15, 16, 19, 20, 22, 23, 24, 27, 28] and references therein. Note that all these papers deal with the subcritical case $N \leq 3$. Recently, Chen and Zou [8] studied the existence and properties of *positive* least energy solutions of (1.2) in the critical case $N = 4$.

On the other hand, there are few results about the existence of sign-changing or semi-nodal solutions to (1.2) in the literature. When $\beta > 0$ is sufficiently large, multiple radially symmetric sign-changing solutions of (1.2) were constructed in [21] for the entire space case. Remark that the method in [21] can not be applied in the non-radial bounded domain case. Recently, the authors [10] proved the existence of infinitely many sign-changing solutions of (1.2) for each fixed $\beta < 0$. Independently, Liu, Liu and Wang [17] obtained infinitely many sign-changing solutions of a general m-coupled system ($m \geq 2$) for each fixed $\beta < 0$. The methods in [10, 17] are completely different.

The main goal of this paper is to study the existence of sign-changing and semi-nodal solutions when $\beta > 0$ is small. This will complement the study made in [10, 17, 21]. Our first result is as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N (N = 2, 3)$ be a smooth bounded domain and $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$. Then for any $k \in \mathbb{N}$ there exists $\beta_k > 0$ such that system (1.2) has at least k sign-changing solutions for each fixed $\beta \in (0, \beta_k)$.*

Definition 1.2. *A nontrivial solution is called a least energy solution, if it has the least energy among all nontrivial solutions. A sign-changing solution is called a least energy sign-changing solution, if it has the least energy among all sign-changing solutions.*

Lin and Wei [16] proved that there exists $\beta_0 > 0$ small such that, for any $\beta \in (-\infty, \beta_0)$, (1.2) has a least energy solution which turns out to be positive. Recently, the existence of least energy sign-changing solutions for $\beta < 0$ was proved in [10]. Here we can prove the following result.

Theorem 1.2. *Let assumptions in Theorem 1.1 hold. Then there exists $\beta_1' \in (0, \beta_1]$ such that system (1.2) has a least energy sign-changing solution for each $\beta \in (0, \beta_1')$.*

Theorems 1.1 and 1.2 are both concerned with sign-changing solutions. The following result is about the existence of multiple semi-nodal solutions.

Theorem 1.3. *Let assumptions in Theorem 1.1 hold. Then for any $k \in \mathbb{N}$ there exists $\beta_k > 0$ such that, for each $\beta \in (0, \beta_k)$, system (1.2) has at least k semi-nodal solutions with the first component sign-changing and the second component positive.*

Remark 1.1. *Similarly we can prove that (1.2) has at least k semi-nodal solutions with the first component positive and the second one sign-changing for each $\beta \in (0, \beta_k)$. Recently, [25, Theorem 0.2] proved the existence of $\beta_k > 0$ such that, for each $\beta \in (0, \beta_k)$, (1.2) has at least k nontrivial solutions $(u_{1,i}, u_{2,i})$ with $u_{1,i} > 0$ in $\Omega\,(i = 1, \cdots, k)$. These solutions are called semi-positive solutions in [25]. Remark that whether $u_{2,i}$ is positive or sign-changing is not known in [25], hence our result improves [25, Theorem 0.2] clearly. Our proofs here are quite different from [25].*

Remark 1.2. *Theorems 1.1-1.3 are all stated in the bounded domain case. Consider the following elliptic system in the entire space:*

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & x \in \mathbb{R}^N, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \mathbb{R}^N, \\ u_1(x),\, u_2(x) \to 0 \ \text{ as } |x| \to +\infty. \end{cases} \tag{1.3}$$

Then by working in the space $H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \,:\, u \text{ radially symmetric}\}$ and recalling the compactness of $H_r^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$, we can prove the following result via the same method: For any $k \in \mathbb{N}$ there exists $\beta_k > 0$ such that, for each fixed $\beta \in (0, \beta_k)$, system (1.3) has at least k radially symmetric sign-changing solutions and k radially symmetric semi-nodal solutions. On the other hand, in 2008 Liu and Wang [18] proved the existence of $\beta_k > 0$ such that, for each

$\beta \in (0, \beta_k)$, (1.3) has at least k nontrivial radially symmetric solutions. In fact, they studied a general m-coupled system ($m \geq 2$). Remark that whether solutions obtained in [18] are positive or sign-changing or semi-nodal is not known. Moreover, Liu and Wang [18, Remark 3.6] suspected that solutions obtained in [18] are not positive solutions, but no proof has yet been given. Our results improve the result of [18] in the two coupled case (m=2).

Remark 1.3. *After the completion of this paper, we learned that (1.3) has also been studied in a recent manuscript [14], where the authors obtained multiple radially symmetric sign-changing solutions with a prescribed number of zeros for $\beta > 0$ small. Remark that their method can not be applied in the non-radial bounded domain case.*

The rest of this paper proves these theorems. We give some notations here. Throughout this paper, we denote the norm of $L^p(\Omega)$ by $|u|_p = (\int_\Omega |u|^p \, dx)^{\frac{1}{p}}$, the norm of $H_0^1(\Omega)$ by $\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2) \, dx$ and positive constants (possibly different in different places) by $C, C_0, C_1, \cdots$. Denote

$$\|u\|_{\lambda_i}^2 := \int_\Omega (|\nabla u|^2 + \lambda_i u^2) \, dx$$

for convenience. Then $\|\cdot\|_{\lambda_i}$ are equivalent norms to $\|\cdot\|$. Define $H := H_0^1(\Omega) \times H_0^1(\Omega)$ with norm $\|(u_1, u_2)\|_H^2 := \|u_1\|_{\lambda_1}^2 + \|u_2\|_{\lambda_2}^2$.

The rest of this paper is organized as follows. In Section 2 we give the proof of Theorem 1.1. The main ideas of our proof are inspired by [10, 26], where a new *constrained problem* introduced by [10] and a new notion of *vector genus* introduced by [26] will be used to define appropriate minimax values. In [26], Tavares and Terracini studied the following general m-coupled system

$$\begin{cases} -\Delta u_j - \mu_j u_j^3 - \beta u_j \sum_{i \neq j} u_i^2 = \lambda_{j,\beta} u_j, \\ u_j \in H_0^1(\Omega), \quad j = 1, \cdots, m, \end{cases} \tag{1.4}$$

where $\beta < 0$, $\mu_j \leq 0$ are all fixed constants. Then [26, Theorem 1.1] says that there exist infinitely many $\lambda = (\lambda_{1,\beta}, \cdots, \lambda_{m,\beta}) \in \mathbb{R}^m$ and $u = (u_1, \cdots, u_m) \in H_0^1(\Omega, \mathbb{R}^m)$ such that (u, λ) are sign-changing solutions of (1.4). That is, $\lambda_{j,\beta}$ is not fixed a *priori* and appears as a Lagrange multiplier in [26]. Here we deal with the focusing case $\mu_j > 0$, and λ_j, μ_j, $\beta > 0$ are all fixed constants. Some arguments in our proof are borrowed from [10, 26] with modifications. Although some procedures are close to those in [10, 26], we prefer to provide all the necessary details to make the paper self-contained. In Section 3 we will use a minimizing argument to prove Theorem 1.2. By giving some modifications to arguments in Sections 2 and 3, we will prove Theorems 1.3 in Section 4.

2 Proof of Theorem 1.1

In the sequel we let assumptions in Theorem 1.1 hold. Without loss of generality we assume that $\mu_1 \geq \mu_2$. Let $\beta \in (0, \mu_2)$. Note that solutions of (1.2) correspond

to the critical points of C^2 functional $E_\beta : H \to \mathbb{R}$ given by

$$E_\beta(u_1,u_2) := \frac{1}{2}\left(\|u_1\|_{\lambda_1}^2 + \|u_2\|_{\lambda_2}^2\right) - \frac{1}{4}\left(\mu_1|u_1|_4^4 + \mu_2|u_2|_4^4\right) - \frac{\beta}{2}\int_\Omega u_1^2u_2^2\,dx. \tag{2.1}$$

Since we are only concerned with nontrivial solutions, we denote $\widetilde{H} := \{(u_1,u_2) \in H : u_i \neq 0 \text{ for } i = 1,2\}$, which is open in H. Write $\vec{u} = (u_1,u_2)$ for convenience.

Lemma 2.1. *For any $\vec{u} = (u_1,u_2) \in \widetilde{H}$, if*

$$\begin{cases} \mu_2|u_2|_4^4\|u_1\|_{\lambda_1}^2 - \beta\|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2u_2^2\,dx > 0, \\ \mu_1|u_1|_4^4\|u_2\|_{\lambda_2}^2 - \beta\|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2u_2^2\,dx > 0, \end{cases} \tag{2.2}$$

then system

$$\begin{cases} \|u_1\|_{\lambda_1}^2 = t_1\mu_1|u_1|_4^4 + t_2\beta\int_\Omega u_1^2u_2^2\,dx \\ \|u_2\|_{\lambda_2}^2 = t_2\mu_2|u_2|_4^4 + t_1\beta\int_\Omega u_1^2u_2^2\,dx \end{cases} \tag{2.3}$$

has a unique solution

$$\begin{cases} t_1(\vec{u}) = \frac{\mu_2|u_2|_4^4\|u_1\|_{\lambda_1}^2 - \beta\|u_2\|_{\lambda_2}^2\int_\Omega u_1^2u_2^2\,dx}{\mu_1\mu_2|u_1|_4^4|u_2|_4^4 - \beta^2(\int_\Omega u_1^2u_2^2\,dx)^2} > 0 \\ t_2(\vec{u}) = \frac{\mu_1|u_1|_4^4\|u_2\|_{\lambda_2}^2 - \beta\|u_1\|_{\lambda_1}^2\int_\Omega u_1^2u_2^2\,dx}{\mu_1\mu_2|u_1|_4^4|u_2|_4^4 - \beta^2(\int_\Omega u_1^2u_2^2\,dx)^2} > 0. \end{cases} \tag{2.4}$$

Moreover,

$$\begin{aligned} \sup_{t_1,t_2\geq 0} E_\beta\left(\sqrt{t_1}u_1,\ \sqrt{t_2}u_2\right) &= E_\beta\left(\sqrt{t_1(\vec{u})}u_1, \sqrt{t_2(\vec{u})}u_2\right) \\ &= \frac{1}{4}\left(t_1(\vec{u})\|u_1\|_{\lambda_1}^2 + t_2(\vec{u})\|u_2\|_{\lambda_2}^2\right) \\ &= \frac{1}{4}\frac{\mu_2|u_2|_4^4\|u_1\|_{\lambda_1}^4 - 2\beta\|u_1\|_{\lambda_1}^2\|u_2\|_{\lambda_2}^2\int_\Omega u_1^2u_2^2\,dx + \mu_1|u_1|_4^4\|u_2\|_{\lambda_2}^4}{\mu_1\mu_2|u_1|_4^4|u_2|_4^4 - \beta^2(\int_\Omega u_1^2u_2^2\,dx)^2} \end{aligned} \tag{2.5}$$

and $(t_1(\vec{u}), t_2(\vec{u}))$ is the unique maximum point of $E_\beta(\sqrt{t_1}u_1, \sqrt{t_2}u_2)$.

Proof. By (2.2) we see that $\mu_1\mu_2|u_1|_4^4|u_2|_4^4 - \beta^2(\int_\Omega u_1^2u_2^2dx)^2 > 0$, so $(t_1(\vec{u}), t_2(\vec{u}))$ defined in (2.4) is the unique solution of (2.3). Note that for $t_1, t_2 \geq 0$,

$$\begin{aligned} f(t_1,t_2) :=& E_\beta\left(\sqrt{t_1}u_1, \sqrt{t_2}u_2\right) = \frac{1}{2}t_1\|u_1\|_{\lambda_1}^2 + \frac{1}{2}t_2\|u_2\|_{\lambda_2}^2 \\ &- \frac{1}{4}\left(t_1^2\mu_1|u_1|_4^4 + t_2^2\mu_2|u_2|_4^4\right) - \frac{1}{2}t_1t_2\beta\int_\Omega u_1^2u_2^2\,dx \\ &\leq \left(\frac{t_1}{2}\|u_1\|_{\lambda_1}^2 - \frac{t_1^2}{4}\mu_1|u_1|_4^4\right) + \left(\frac{t_2}{2}\|u_2\|_{\lambda_2}^2 - \frac{t_2^2}{4}\mu_2|u_2|_4^4\right). \end{aligned}$$

This implies that $f(t_1,t_2) < 0$ for $\max\{t_1,t_2\} > T$, where T is some positive constant. So there exists $(\tilde{t}_1, \tilde{t}_2) \in [0,T]^2 \setminus \{(0,0)\}$ such that

$$f(\tilde{t}_1, \tilde{t}_2) = \sup_{t_1,t_2\geq 0} f(t_1,t_2).$$

It suffices to show that $(\tilde{t}_1, \tilde{t}_2) = (t_1(\vec{u}), t_2(\vec{u}))$. Note that

$$\sup_{t_1 \geq 0} f(t_1, 0) = \frac{1}{4} \frac{\|u_1\|_{\lambda_1}^4}{\mu_1 |u_1|_4^4}.$$

Recalling the expression of $f(t_1(\vec{u}), t_2(\vec{u}))$ in (2.5), by a direct computation we deduce from (2.2) that

$$f(t_1(\vec{u}), t_2(\vec{u})) - \sup_{t_1 \geq 0} f(t_1, 0) = \frac{(\mu_1 |u_1|_4^4 \|u_2\|_{\lambda_2}^2 - \beta \|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2 u_2^2 \, dx)^2}{4\mu_1 |u_1|_4^4 [\mu_1 \mu_2 |u_1|_4^4 |u_2|_4^4 - \beta^2 (\int_\Omega u_1^2 u_2^2 \, dx)^2]} > 0.$$

Similarly we have $f(t_1(\vec{u}), t_2(\vec{u})) - \sup_{t_2 \geq 0} f(0, t_2) > 0$, so $\tilde{t}_1 > 0$ and $\tilde{t}_2 > 0$. Then by $\frac{\partial}{\partial t_1} f(t_1, t_2)|_{(\tilde{t}_1, \tilde{t}_2)} = \frac{\partial}{\partial t_2} f(t_1, t_2)|_{(\tilde{t}_1, \tilde{t}_2)} = 0$ we see that $(\tilde{t}_1, \tilde{t}_2)$ satisfies (2.3), so $(\tilde{t}_1, \tilde{t}_2) = (t_1(\vec{u}), t_2(\vec{u}))$. $\square$

Define

$$\mathcal{M}^* := \left\{ \vec{u} \in H \ : \ 1/2 < |u_1|_4^4 < 2, \ 1/2 < |u_2|_4^4 < 2 \right\}; \tag{2.6}$$

$$\mathcal{M}_\beta^* := \{ \vec{u} \in \mathcal{M}^* \ : \ \vec{u} \text{ satisfies (2.2)} \};$$

$$\mathcal{M}_\beta^{**} := \left\{ \vec{u} \in \mathcal{M}^* \ : \ \begin{array}{l} \mu_2 \|u_1\|_{\lambda_1}^2 - \beta \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2 \, dx > 0 \\ \mu_1 \|u_2\|_{\lambda_2}^2 - \beta \|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2 u_2^2 \, dx > 0 \end{array} \right\};$$

$$\mathcal{M} := \{ \vec{u} \in H \ : \ |u_1|_4 = 1, \ |u_2|_4 = 1 \}, \quad \mathcal{M}_\beta := \mathcal{M} \cap \mathcal{M}_\beta^*. \tag{2.7}$$

Then $\mathcal{M}_\beta = \mathcal{M} \cap \mathcal{M}_\beta^{**}$. Evidently $\mathcal{M}^*$, $\mathcal{M}_\beta^*$, $\mathcal{M}_\beta^{**}$ are all open subsets of H and $\mathcal{M}$ is closed. Note that $\mu_1 \mu_2 - \beta^2 (\int_\Omega u_1^2 u_2^2 \, dx)^2 > 0$ for any $\vec{u} \in \mathcal{M}_\beta^{**}$, as in [10] we define a new functional $J_\beta : \mathcal{M}_\beta^{**} \to (0, +\infty)$ by

$$J_\beta(\vec{u}) := \frac{1}{4} \frac{\mu_2 \|u_1\|_{\lambda_1}^4 - 2\beta \|u_1\|_{\lambda_1}^2 \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2 \, dx + \mu_1 \|u_2\|_{\lambda_2}^4}{\mu_1 \mu_2 - \beta^2 (\int_\Omega u_1^2 u_2^2 \, dx)^2}.$$

A direct computation yields $J_\beta \in C^1(\mathcal{M}_\beta^{**}, (0, +\infty))$. Moreover, since any $\vec{u} \in \mathcal{M}_\beta$ is an interior point of $\mathcal{M}_\beta^{**}$, by (2.4) we can prove that

$$J_\beta'(\vec{u})(\varphi, 0) = t_1(\vec{u}) \int_\Omega (\nabla u_1 \nabla \varphi + \lambda_1 u_1 \varphi) \, dx - t_1(\vec{u}) t_2(\vec{u}) \beta \int_\Omega u_1 u_2^2 \varphi \, dx, \tag{2.8}$$

$$J_\beta'(\vec{u})(0, \psi) = t_2(\vec{u}) \int_\Omega (\nabla u_2 \nabla \psi + \lambda_2 u_2 \psi) \, dx - t_1(\vec{u}) t_2(\vec{u}) \beta \int_\Omega u_1^2 u_2 \psi \, dx \tag{2.9}$$

hold for any $\vec{u} \in \mathcal{M}_\beta$ and φ, $\psi \in H_0^1(\Omega)$ (Remark that (2.8)-(2.9) do not hold for $\vec{u} \in \mathcal{M}_\beta^{**} \setminus \mathcal{M}_\beta$). Note that Lemma 2.1 yields

$$J_\beta(u_1, u_2) = \sup_{t_1, t_2 \geq 0} E_\beta \left(\sqrt{t_1} u_1, \sqrt{t_2} u_2 \right), \quad \forall \, (u_1, u_2) \in \mathcal{M}_\beta. \tag{2.10}$$

To obtain nontrivial solutions of (1.2), we turn to study the functional J_β restricted to $\mathcal{M}_\beta$, which is a problem with two constraints. Define

$$\mathcal{N}_b^* := \left\{ \vec{u} \in \mathcal{M}^* \ : \ \|u_1\|_{\lambda_1}^2, \|u_2\|_{\lambda_2}^2 < b \right\}, \quad \mathcal{N}_b := \mathcal{N}_b^* \cap \mathcal{M}. \tag{2.11}$$

Fix any $k \in \mathbb{N}$. Our goal is to prove the existence of $\beta_k > 0$ such that (1.2) has at least k sign-changing solutions for any $\beta \in (0, \beta_k)$. To do this, we let W_{k+1} be a $k+1$ dimensional subspace of $H_0^1(\Omega)$ which contains an element φ_0 satisfying $\varphi_0 > 0$ in Ω. Then we can find $\bar{b} > 0$ such that

$$\|u\|_{\lambda_1}^2,\ \|u\|_{\lambda_2}^2 < \bar{b}, \quad \forall\, u \in W_{k+1} \text{ satisfying } |u|_4^4 < 2. \tag{2.12}$$

Fix a $b > 0$ such that

$$b^2 > (2 + \mu_1/\mu_2)\bar{b}^2. \tag{2.13}$$

Then $\mathcal{N}_{\bar{b}}^* \subset \mathcal{N}_b^*$ and $\mathcal{N}_{\bar{b}} \subset \mathcal{N}_b$. Recalling the Sobolev inequality

$$\|u\|_{\lambda_i}^2 \geq S|u|_4^2, \quad \forall\, u \in H_0^1(\Omega),\ \ i = 1, 2, \tag{2.14}$$

where S is a positive constant, we have the following lemma.

Lemma 2.2. *There exist $\beta_0 \in (0, \mu_2)$ and $C_1 > C_0 > 0$ such that for any $\beta \in (0, \beta_0)$ there hold $\mathcal{N}_b^* \subset \mathcal{M}_\beta^* \cap \mathcal{M}_\beta^{**}$ and*

$$C_0 \leq t_1(\vec{u}),\, t_2(\vec{u}) \leq C_1, \quad \forall\, \vec{u} \in \mathcal{N}_b^*.$$

Proof. Define $\beta_0 := \frac{\mu_2 S}{8b}$ and let $\beta \in (0, \beta_0)$. For any $\vec{u} = (u_1, u_2) \in \mathcal{N}_b^*$, we see from (2.6) and (2.14) that $\int_\Omega u_1^2 u_2^2\, dx \leq |u_1|_4^2 |u_2|_4^2 < 2$ and $\|u_i\|_{\lambda_i}^2 \geq S/\sqrt{2}$. Hence

$$\begin{aligned}
&\mu_2 |u_2|_4^4 \|u_1\|_{\lambda_1}^2 - \beta \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2\, dx \geq \frac{\mu_2 S}{2\sqrt{2}} - 2b\beta_0 \geq \frac{\mu_2 S}{16};\\
&\mu_1 |u_1|_4^4 \|u_2\|_{\lambda_2}^2 - \beta \|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2 u_2^2\, dx \geq \frac{\mu_2 S}{16};\\
&\mu_2 \|u_1\|_{\lambda_1}^2 - \beta \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2\, dx \geq \frac{\mu_2 S}{16};\\
&\mu_1 \|u_2\|_{\lambda_2}^2 - \beta \|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2 u_2^2\, dx \geq \frac{\mu_2 S}{16};\\
&\mu_1 \mu_2 - \beta^2 \left(\int_\Omega u_1^2 u_2^2\, dx \right)^2 \geq \frac{\mu_2^2 S^2}{2^8} \cdot \frac{1}{\|u_1\|_{\lambda_1}^2 \|u_2\|_{\lambda_2}^2} \geq \frac{\mu_2^2 S^2}{2^8 b^2};\\
&\mu_1 \mu_2 |u_1|_4^4 |u_2|_4^4 - \beta^2 \left(\int_\Omega u_1^2 u_2^2\, dx \right)^2 \geq \frac{\mu_2^2 S^2}{2^8 b^2}.
\end{aligned}$$

Then $\vec{u} \in \mathcal{M}_\beta^* \cap \mathcal{M}_\beta^{**}$. Moreover, combining these with (2.4) we have

$$t_i(\vec{u}) \geq \frac{\mu_2 S}{2^4} \cdot \frac{1}{\mu_1 \mu_2 |u_1|_4^4 |u_2|_4^4} \geq \frac{S}{2^6 \mu_1},\ \ t_i(\vec{u}) \leq \frac{2^9 b^3}{\mu_2^2 S^2} \mu_1,\ \ i = 1, 2.$$

This completes the proof. □

Lemma 2.3. *There exist $\beta_k \in (0, \beta_0]$ and $d_k > 0$ such that*

$$\inf_{\vec{u} \in \partial \mathcal{N}_b} J_\beta(\vec{u}) \geq d_k > \sup_{\vec{u} \in \mathcal{N}_{\bar{b}}} J_\beta(\vec{u}), \quad \forall\, \beta \in (0, \beta_k). \tag{2.15}$$

Proof. This proof is inspired by [25]. Define

$$I_i(u_i) := \frac{1}{4\mu_i}\|u_i\|_{\lambda_i}^4, \quad i=1,2.$$

Then for any $\vec{u} \in \overline{\mathcal{N}_b}$ and $\beta \in (0, \beta_0)$ we have

$$\begin{aligned}
&|J_\beta(\vec{u}) - I_1(u_1) - I_2(u_2)| \\
=&\frac{\beta\left|\beta(\int_\Omega u_1^2u_2^2\,dx)^2\sum_{i=1}^2\|u_i\|_{\lambda_i}^4/\mu_i - 2\|u_1\|_{\lambda_1}^2\|u_2\|_{\lambda_2}^2\int_\Omega u_1^2u_2^2\,dx\right|}{4[\mu_1\mu_2 - \beta^2(\int_\Omega u_1^2u_2^2\,dx)^2]} \le C\beta,
\end{aligned}$$

where $C > 0$ is independent of $\vec{u} \in \overline{\mathcal{N}_b}$ and $\beta \in (0, \beta_0)$. Therefore,

$$\begin{aligned}
&\sup_{\vec{u}\in\mathcal{N}_{\bar{b}}} J_\beta(\vec{u}) \le \sup_{\vec{u}\in\mathcal{N}_{\bar{b}}}(I_1(u_1)+I_2(u_2)) + C\beta \le \frac{\bar{b}^2}{4\mu_1} + \frac{\bar{b}^2}{4\mu_2} + C\beta; \\
&\inf_{\vec{u}\in\partial\mathcal{N}_b} J_\beta(\vec{u}) \ge \inf_{\vec{u}\in\partial\mathcal{N}_b}(I_1(u_1)+I_2(u_2)) - C\beta \ge \frac{b^2}{4\mu_1} - C\beta.
\end{aligned}$$

Recalling (2.13), we let $\beta_k = \min\{\frac{\bar{b}^2}{8\mu_1 C}, \beta_0\}$ and $d_k = \frac{b^2}{4\mu_1} - C\beta_k$, then (2.15) holds. This completes the proof. $\square$

In the following we always let $(i,j) = (1,2)$ or $(i,j) = (2,1)$. Recalling (2.14) and Lemma 2.2, we can take β_k smaller if necessary such that, for any $\beta \in (0, \beta_k)$ and $\vec{u} \in \mathcal{N}_b^*$, there holds

$$\|v\|_{\lambda_i}^2 - \beta t_j(\vec{u})\int_\Omega u_j^2 v^2\,dx \ge \frac{1}{2}\|v\|_{\lambda_i}^2, \quad \forall\, v \in H_0^1(\Omega),\ i=1,2. \tag{2.16}$$

Clearly (2.16) implies that the operators $-\Delta+\lambda_i-\beta t_j(\vec{u})u_j^2$ are positive definite in $H_0^1(\Omega)$. In the rest of this section we fix any $\beta \in (0, \beta_k)$. We will show that (1.2) has at least k sign-changing solutions. For any $\vec{u} = (u_1, u_2) \in \mathcal{N}_b^*$, let $\tilde{w}_i \in H_0^1(\Omega)$ be the unique solution of the following linear problem

$$-\Delta\tilde{w}_i + \lambda_i\tilde{w}_i - \beta t_j(\vec{u})u_j^2\tilde{w}_i = \mu_i t_i(\vec{u})u_i^3, \quad \tilde{w}_i \in H_0^1(\Omega). \tag{2.17}$$

Since $|u_i|_4^4 > 1/2$, so $\tilde{w}_i \ne 0$ and we see from (2.16) that

$$\int_\Omega u_i^3\tilde{w}_i\,dx = \frac{1}{\mu_i t_i(\vec{u})}\left(\|\tilde{w}_i\|_{\lambda_i}^2 - \beta t_j(\vec{u})\int_\Omega u_j^2\tilde{w}_i^2\,dx\right) \ge \frac{1}{2\mu_i t_i(\vec{u})}\|\tilde{w}_i\|_{\lambda_i}^2 > 0.$$

Define

$$w_i = \alpha_i\tilde{w}_i, \quad \text{where } \alpha_i = \frac{1}{\int_\Omega u_i^3\tilde{w}_i\,dx} > 0. \tag{2.18}$$

Then w_i is the unique solution of the following problem

$$\begin{cases} -\Delta w_i + \lambda_i w_i - \beta t_j(\vec{u})u_j^2 w_i = \alpha_i\mu_i t_i(\vec{u})u_i^3, \quad w_i \in H_0^1(\Omega), \\ \int_\Omega u_i^3 w_i\,dx = 1. \end{cases} \tag{2.19}$$

Now we define an operator $K = (K_1, K_2) : \mathcal{N}_b^* \to H$ by

$$K(\vec{u}) = (K_1(\vec{u}), K_2(\vec{u})) := \vec{w} = (w_1, w_2). \tag{2.20}$$

Define the transformations

$$\sigma_i : H \to H \quad \text{by} \quad \sigma_1(u_1, u_2) := (-u_1, u_2), \;\; \sigma_2(u_1, u_2) := (u_1, -u_2). \tag{2.21}$$

Then it is easy to check that

$$K(\sigma_i(\vec{u})) = \sigma_i(K(\vec{u})), \quad i = 1, 2. \tag{2.22}$$

Lemma 2.4. $K \in C^1(\mathcal{N}_b^*, H)$.

Proof. It suffices to apply the Implicit Theorem to the C^1 map

$$\Psi : \mathcal{N}_b^* \times H_0^1(\Omega) \times \mathbb{R} \to H_0^1(\Omega) \times \mathbb{R}, \quad \text{where}$$
$$\Psi(\vec{u}, v, \alpha) = \left(v - (-\Delta + \lambda_i)^{-1} \left(\beta t_j(\vec{u}) u_j^2 v + \alpha \mu_i t_i(\vec{u}) u_i^3 \right), \; \int_\Omega u_i^3 v \, dx - 1 \right).$$

Note that (2.19) holds if and only if $\Psi(\vec{u}, w_i, \alpha_i) = (0, 0)$. By computing the derivative of Ψ with respect to (v, α) at the point $(\vec{u}, w_i, \alpha_i)$ in the direction $(\bar{w}, \bar{\alpha})$, we obtain a map $\Phi : H_0^1(\Omega) \times \mathbb{R} \to H_0^1(\Omega) \times \mathbb{R}$ given by

$$\begin{aligned} \Phi(\bar{w}, \bar{\alpha}) :=& D_{v,\alpha} \Psi(\vec{u}, w_i, \alpha_i)(\bar{w}, \bar{\alpha}) \\ =& \left(\bar{w} - (-\Delta + \lambda_i)^{-1} \left(\beta t_j(\vec{u}) u_j^2 \bar{w} + \bar{\alpha} \mu_i t_i(\vec{u}) u_i^3 \right), \; \int_\Omega u_i^3 \bar{w} \, dx \right). \end{aligned}$$

Recalling (2.16), similarly as [10, Lemma 2.3] it is easy to prove that Φ is a bijective map. We omit the details. $\square$

Lemma 2.5. *Assume that $\{\vec{u}_n = (u_{n,1}, u_{n,2}) : n \geq 1\} \subset \mathcal{N}_b$. Then there exists $\vec{w} \in H$ such that, up to a subsequence, $\vec{w}_n := K(\vec{u}_n) \to \vec{w}$ strongly in H.*

Proof. Up to a subsequence, we may assume that $\vec{u}_n \rightharpoonup \vec{u} = (u_1, u_2)$ weakly in H and so $u_{n,i} \to u_i$ strongly in $L^4(\Omega)$, which implies $|u_i|_4 = 1$. Moreover, by Lemma 2.2 we may assume $t_i(\vec{u}_n) \to t_i > 0$. Recall that $w_{n,i} = \alpha_{n,i} \tilde{w}_{n,i}$, where $\alpha_{n,i}$ and $\tilde{w}_{n,i}$ are seen in (2.17)-(2.18). By (2.16)-(2.17) we have

$$\frac{1}{2} \|\tilde{w}_{n,i}\|_{\lambda_i}^2 \leq \mu_i t_i(\vec{u}_n) \int_\Omega u_{n,i}^3 \tilde{w}_{n,i} \, dx \leq C |\tilde{w}_{n,i}|_4 \leq C \|\tilde{w}_{n,i}\|_{\lambda_i},$$

which implies that $\{\tilde{w}_{n,i} : n \geq 1\}$ are bounded in $H_0^1(\Omega)$. Up to a subsequence, we may assume that $\tilde{w}_{n,i} \to \tilde{w}_i$ weakly in $H_0^1(\Omega)$ and strongly in $L^4(\Omega)$. Then by (2.17) it is standard to prove that $\tilde{w}_{n,i} \to \tilde{w}_i$ strongly in $H_0^1(\Omega)$. Moreover, $\tilde{w}_i$ satisfies $-\Delta \tilde{w}_i + \lambda_i \tilde{w}_i - \beta t_j u_j^2 \tilde{w}_i = \mu_i t_i u_i^3$. Since $|u_i|_4 = 1$, so $\tilde{w}_i \neq 0$ and then $\int_\Omega u_i^3 \tilde{w}_i \, dx > 0$, which implies that

$$\lim_{n \to \infty} \alpha_{n,i} = \lim_{n \to \infty} \frac{1}{\int_\Omega u_{n,i}^3 \tilde{w}_{n,i} \, dx} = \frac{1}{\int_\Omega u_i^3 \tilde{w}_i \, dx} =: \alpha_i.$$

Therefore, $w_{n,i} = \alpha_{n,i}\tilde{w}_{n,i} \to \alpha_i \tilde{w}_i =: w_i$ strongly in $H_0^1(\Omega)$. $\square$

To continue our proof, we need to use *vector genus* introduced by [26] to define proper minimax energy levels. Recall (2.7) and (2.21), as in [26] we consider the class of sets

$$\mathcal{F} = \{A \subset \mathcal{M} : A \text{ is closed and } \sigma_i(\vec{u}) \in A \ \forall \vec{u} \in A,\ i = 1, 2\},$$

and, for each $A \in \mathcal{F}$ and $k_1, k_2 \in \mathbb{N}$, the class of functions

$$F_{(k_1,k_2)}(A) = \left\{ f = (f_1, f_2) : A \to \prod_{i=1}^{2} \mathbb{R}^{k_i - 1} : \begin{array}{l} f_i : A \to \mathbb{R}^{k_i-1} \text{ continuous,} \\ f_i(\sigma_i(\vec{u})) = -f_i(\vec{u}) \text{ for each } i, \\ f_i(\sigma_j(\vec{u})) = f_i(\vec{u}) \text{ for } j \neq i \end{array} \right\}.$$

Here we denote $\mathbb{R}^0 := \{0\}$. Let us recall vector genus from [26].

Definition 2.1. *(Vector genus, see [26]) Let $A \in \mathcal{F}$ and take any $k_1, k_2 \in \mathbb{N}$. We say that $\vec{\gamma}(A) \geq (k_1, k_2)$ if for every $f \in F_{(k_1,k_2)}(A)$ there exists $\vec{u} \in A$ such that $f(\vec{u}) = (f_1(\vec{u}), f_2(\vec{u})) = (0, 0)$. We denote*

$$\Gamma^{(k_1,k_2)} := \{A \in \mathcal{F} : \vec{\gamma}(A) \geq (k_1, k_2)\}.$$

Lemma 2.6. *(see [26]) With the previous notations, the following properties hold.*

- (i) *Take $A_1 \times A_2 \subset \mathcal{M}$ and let $\eta_i : S^{k_i-1} := \{x \in \mathbb{R}^{k_i} : |x| = 1\} \to A_i$ be a homeomorphism such that $\eta_i(-x) = -\eta_i(x)$ for every $x \in S^{k_i-1}$, $i = 1, 2$. Then $A_1 \times A_2 \in \Gamma^{(k_1,k_2)}$.*
- (ii) *We have $\overline{\eta(A)} \in \Gamma^{(k_1,k_2)}$ whenever $A \in \Gamma^{(k_1,k_2)}$ and a continuous map $\eta : A \to \mathcal{M}$ is such that $\eta \circ \sigma_i = \sigma_i \circ \eta,\ \forall i = 1, 2$.*

To obtain sign-changing solutions, as in many references such as [11, 4, 29], we should use cones of positive functions. Precisely, we define

$$\mathcal{P}_i := \{\vec{u} = (u_1, u_2) \in H : u_i \geq 0\}, \quad \mathcal{P} := \bigcup_{i=1}^{2} (\mathcal{P}_i \cup -\mathcal{P}_i). \tag{2.23}$$

Moreover, for $\delta > 0$ we define $\mathcal{P}_\delta := \{\vec{u} \in H : \mathrm{dist}_4(\vec{u}, \mathcal{P}) < \delta\}$, where

$$\begin{aligned} &\mathrm{dist}_4(\vec{u}, \mathcal{P}) := \min\left\{\mathrm{dist}_4(u_i, \mathcal{P}_i),\ \mathrm{dist}_4(u_i, -\mathcal{P}_i), \quad i = 1, 2\right\}, \\ &\mathrm{dist}_4(u_i, \pm\mathcal{P}_i) := \inf\{|u_i - v|_4 \ :\ v \in \pm\mathcal{P}_i\}. \end{aligned} \tag{2.24}$$

Denote $u^\pm := \max\{0, \pm u\}$, then it is easy to check that $\mathrm{dist}_4(u_i, \pm\mathcal{P}_i) = |u_i^\mp|_4$. The following lemma was proved in [10].

Lemma 2.7. *(see [10, Lemma 2.6]) Let $k_1, k_2 \geq 2$. Then for any $\delta < 2^{-1/4}$ and any $A \in \Gamma^{(k_1,k_2)}$ there holds $A \setminus \mathcal{P}_\delta \neq \emptyset$.*

Lemma 2.8. *There exists $A \in \Gamma^{(k+1,k+1)}$ such that $A \subset \mathcal{N}_b$ and $\sup_A J_\beta < d_k$.*

Proof. Recalling W_{k+1} in (2.12), we define

$$A_1 = A_2 := \big\{u \in W_{k+1} \,:\, |u|_4 = 1\big\}.$$

Note that there exists an obvious odd homeomorphism from S^k to A_i. By Lemma 2.6-(i) one has $A := A_1 \times A_2 \in \Gamma^{(k+1,k+1)}$. We see from (2.12) that $A \subset \mathcal{N}_{\bar{b}}$, and so Lemma 2.3 yields $\sup_A J_\beta < d_k$. $\square$

For every $k_1, k_2 \in [2, k+1]$ and $0 < \delta < 2^{-1/4}$, we define

$$c_{\beta,\delta}^{k_1,k_2} := \inf_{A \in \Gamma_\beta^{(k_1,k_2)}} \sup_{\vec{u} \in A \setminus \mathcal{P}_\delta} J_\beta(\vec{u}), \tag{2.25}$$

where

$$\Gamma_\beta^{(k_1,k_2)} := \left\{A \in \Gamma^{(k_1,k_2)} \,:\, A \subset \mathcal{N}_b,\ \sup_A J_\beta < d_k\right\}. \tag{2.26}$$

Noting that $\Gamma_\beta^{(\tilde{k}_1,\tilde{k}_2)} \subset \Gamma_\beta^{(k_1,k_2)}$ for any $\tilde{k}_1 \ge k_1$ and $\tilde{k}_2 \ge k_2$, we see that Lemma 2.8 yields $\Gamma_\beta^{(k_1,k_2)} \neq \emptyset$ and so $c_{\beta,\delta}^{k_1,k_2}$ is well defined for any $k_1, k_2 \in [2, k+1]$. Moreover,

$$c_{\beta,\delta}^{k_1,k_2} < d_k \quad \text{for every } \delta \in (0, 2^{-1/4}) \text{ and } k_1, k_2 \in [2, k+1].$$

We will prove that $c_{\beta,\delta}^{k_1,k_2}$ is a critical value of E_β for $\delta > 0$ sufficiently small. Define $\mathcal{N}_{b,\beta} := \{\vec{u} \in \mathcal{N}_b : J_\beta(\vec{u}) < d_k\}$, then Lemma 2.3 yields $\mathcal{N}_{\bar{b}} \subset \mathcal{N}_{b,\beta}$.

Lemma 2.9. *For any sufficiently small $\delta \in (0, 2^{-1/4})$, there holds*

$$dist_4(K(\vec{u}), \mathcal{P}) < \delta/2, \quad \forall\, \vec{u} \in \mathcal{N}_{b,\beta},\ dist_4(\vec{u}, \mathcal{P}) < \delta.$$

Proof. Assume by contradiction that there exist $\delta_n \to 0$ and $\vec{u}_n = (u_{n,1}, u_{n,2}) \in \mathcal{N}_{b,\beta}$ such that $\mathrm{dist}_4(\vec{u}_n, \mathcal{P}) < \delta_n$ and $\mathrm{dist}_4(K(\vec{u}_n), \mathcal{P}) \ge \delta_n/2$. Without loss of generality we may assume that $\mathrm{dist}_4(\vec{u}_n, \mathcal{P}) = \mathrm{dist}_4(u_{n,1}, \mathcal{P}_1)$. Write $K(\vec{u}_n) = \vec{w}_n = (w_{n,1}, w_{n,2})$ and $w_{n,i} = \alpha_{n,i}\tilde{w}_{n,i}$ as in Lemma 2.5. Then by the proof of Lemma 2.5, we see that $\alpha_{n,i}$ are all uniformly bounded. Combining this with (2.16) and (2.19), we deduce that

$$\begin{aligned}
\mathrm{dist}_4(w_{n,1}, \mathcal{P}_1)|w_{n,1}^-|_4 &= |w_{n,1}^-|_4^2 \le C\|w_{n,1}^-\|_{\lambda_1}^2 \\
&\le C\int_\Omega \left(|\nabla w_{n,1}^-|^2 + \lambda_1 (w_{n,1}^-)^2 - \beta t_2(\vec{u}_n) u_{n,2}^2 (w_{n,1}^-)^2\right) dx \\
&= -C\alpha_{n,1}\mu_1 t_1(\vec{u}_n) \int_\Omega u_{n,1}^3 w_{n,1}^- \, dx \\
&\le C\int_\Omega (u_{n,1}^-)^3 w_{n,1}^- \, dx \le C|u_{n,1}^-|_4^3 |w_{n,1}^-|_4 \\
&= C\mathrm{dist}_4(u_{n,1}, \mathcal{P}_1)^3 |w_{n,1}^-|_4 \le C\delta_n^3 |w_{n,1}^-|_4.
\end{aligned}$$

So $\mathrm{dist}_4(K(\vec{u}_n), \mathcal{P}) \leq \mathrm{dist}_4(w_{n,1}, \mathcal{P}_1) \leq C\delta_n^3 < \delta_n/2$ holds for n sufficiently large, which is a contradiction. This completes the proof. $\square$

Now let us define a map $V : \mathcal{N}_b^* \to H$ by $V(\vec{u}) := \vec{u} - K(\vec{u})$. We will prove that $(\sqrt{t_1(\vec{u})}u_1, \sqrt{t_2(\vec{u})}u_2)$ is a sign-changing solution of (1.2) if $\vec{u} = (u_1, u_2) \in \mathcal{N}_b \setminus \mathcal{P}$ satisfies $V(\vec{u}) = 0$.

Lemma 2.10. *Let $\vec{u}_n = (u_{n,1}, u_{n,2}) \in \mathcal{N}_b$ be such that*

$$J_\beta(\vec{u}_n) \to c < d_k \quad \textit{and} \quad V(\vec{u}_n) \to 0 \quad \textit{strongly in } H.$$

Then up to a subsequence, there exists $\vec{u} \in \mathcal{N}_b$ such that $\vec{u}_n \to \vec{u}$ strongly in H and $V(\vec{u}) = 0$.

Proof. By Lemma 2.5, up to a subsequence, we may assume that $\vec{u}_n \rightharpoonup \vec{u} = (u_1, u_2)$ weakly in H and $\vec{w}_n := K(\vec{u}_n) = (w_{n,1}, w_{n,2}) \to \vec{w} = (w_1, w_2)$ strongly in H. Recalling $V(\vec{u}_n) \to 0$, we get

$$\begin{aligned}&\int_\Omega \nabla u_{n,i} \nabla (u_{n,i} - u_i)\, dx = \int_\Omega \nabla (w_{n,i} - w_i) \nabla (u_{n,i} - u_i)\, dx \\ &+ \int_\Omega \nabla w_i \nabla (u_{n,i} - u_i)\, dx + \int_\Omega \nabla (u_{n,i} - w_{n,i}) \nabla (u_{n,i} - u_i)\, dx = o(1).\end{aligned}$$

Then it is easy to see that $\vec{u}_n \to \vec{u}$ strongly in H and so $\vec{u} \in \overline{\mathcal{N}_b}$. Hence $V(\vec{u}) = \lim_{n\to\infty} V(\vec{u}_n) = 0$. Moreover, $J_\beta(\vec{u}) = c < d_k$ and so $\vec{u} \in \mathcal{N}_b$. $\square$

Lemma 2.11. *Recall $C_0 > 0$ in Lemma 2.2. Then*

$$J'_\beta(\vec{u})[V(\vec{u})] \geq \frac{C_0}{2} \|V(\vec{u})\|_H^2, \quad \textit{for any } \vec{u} \in \mathcal{N}_b.$$

Proof. Fix any $\vec{u} = (u_1, u_2) \in \mathcal{N}_b$ and write $\vec{w} = K(\vec{u}) = (w_1, w_2)$ as above, then $V(\vec{u}) = (u_1 - w_1, u_2 - w_2)$. By (2.19) we have $\int_\Omega u_i^3 (u_i - w_i)\, dx = 1 - 1 = 0$. Then we deduce from (2.8)-(2.9), (2.16) and (2.19) that

$$\begin{aligned}&J'_\beta(\vec{u})[V(\vec{u})] \\ =&\sum_{i=1}^2 t_i(\vec{u}) \int_\Omega \left(\nabla u_i \nabla (u_i - w_i) + \lambda_i u_i (u_i - w_i) - t_j(\vec{u})\beta u_i (u_i - w_i) u_j^2\right) dx \\ =&\sum_{i=1}^2 t_i(\vec{u}) \int_\Omega \big(\nabla u_i \nabla (u_i - w_i) + \lambda_i u_i (u_i - w_i) - t_j(\vec{u})\beta w_i (u_i - w_i) u_j^2 \\ &\qquad\qquad - t_j(\vec{u})\beta (u_i - w_i)^2 u_j^2\big)\, dx \\ =&\sum_{i=1}^2 t_i(\vec{u}) \int_\Omega \big(\nabla u_i \nabla (u_i - w_i) + \lambda_i u_i (u_i - w_i) - \nabla w_i \nabla (u_i - w_i) \\ &\quad - \lambda_i w_i (u_i - w_i) + \alpha_i \mu_i t_i(\vec{u}) u_i^3 (u_i - w_i) - t_j(\vec{u})\beta (u_i - w_i)^2 u_j^2\big)\, dx\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} t_i(\vec{u}) \int_\Omega \left(|\nabla(u_i - w_i)|^2 + \lambda_i |u_i - w_i|^2 - t_j(\vec{u})\beta(u_i - w_i)^2 u_j^2 \right) dx \\
&\geq \sum_{i=1}^{2} \frac{t_i(\vec{u})}{2} \|u_i - w_i\|_{\lambda_i}^2 \geq \frac{C_0}{2} \|V(\vec{u})\|_H^2.
\end{aligned}
$$

This completes the proof. □

Lemma 2.12. *There exists a unique global solution $\eta = (\eta_1, \eta_2) : [0, \infty) \times \mathcal{N}_{b,\beta} \to H$ for the initial value problem*

$$
\frac{d}{dt}\eta(t, \vec{u}) = -V(\eta(t, \vec{u})), \quad \eta(0, \vec{u}) = \vec{u} \in \mathcal{N}_{b,\beta}. \tag{2.27}
$$

Moreover,

(i) $\eta(t, \vec{u}) \in \mathcal{N}_{b,\beta}$ for any $t > 0$ and $\vec{u} \in \mathcal{N}_{b,\beta}$.

(ii) $\eta(t, \sigma_i(\vec{u})) = \sigma_i(\eta(t, \vec{u}))$ for any $t > 0$, $\vec{u} \in \mathcal{N}_{b,\beta}$ and $i = 1, 2$.

(iii) For every $\vec{u} \in \mathcal{N}_{b,\beta}$, the map $t \mapsto J_\beta(\eta(t, \vec{u}))$ is non-increasing.

(iv) There exists $\delta_0 \in (0, 2^{-1/4})$ such that, for every $\delta < \delta_0$, there holds

$$
\eta(t, \vec{u}) \in \mathcal{P}_\delta \quad \textit{whenever } \vec{u} \in \mathcal{N}_{b,\beta} \cap \mathcal{P}_\delta \textit{ and } t > 0.
$$

Proof. Recalling Lemma 2.4, we have $V(\vec{u}) \in C^1(\mathcal{N}_b^*, H)$. Since $\mathcal{N}_{b,\beta} \subset \mathcal{N}_b^*$ and $\mathcal{N}_b^*$ is open, so (2.27) has a unique solution $\eta : [0, T_{\max}) \times \mathcal{N}_{b,\beta} \to H$, where $T_{\max} > 0$ is the maximal time such that $\eta(t, \vec{u}) \in \mathcal{N}_b^*$ for all $t \in [0, T_{\max})$ (Note that $V(\cdot)$ is defined only on $\mathcal{N}_b^*$). We should prove $T_{\max} = +\infty$ for any $\vec{u} \in \mathcal{N}_{b,\beta}$. Fixing any $\vec{u} = (u_1, u_2) \in \mathcal{N}_{b,\beta}$, we have

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \eta_i(t, \vec{u})^4 \, dx &= -4 \int_\Omega \eta_i(t, \vec{u})^3 (\eta_i(t, \vec{u}) - K_i(\eta(t, \vec{u}))) \, dx \\
&= 4 - 4 \int_\Omega \eta_i(t, \vec{u})^4 \, dx, \quad \forall\, 0 < t < T_{\max}.
\end{aligned}
$$

Recalling $\int_\Omega \eta_i(0, \vec{u})^4 \, dx = \int_\Omega u_i^4 \, dx = 1$, we deduce that $\int_\Omega \eta_i(t, \vec{u})^4 \, dx \equiv 1$ for all $0 \leq t < T_{\max}$. So $\eta(t, \vec{u}) \in \mathcal{M}$, that is $\eta(t, \vec{u}) \in \mathcal{M} \cap \mathcal{N}_b^* = \mathcal{N}_b$ for all $t \in [0, T_{\max})$. Assume by contradiction that $T_{\max} < +\infty$, then $\eta(T_{\max}, \vec{u}) \in \partial\mathcal{N}_b$, and so $J_\beta(\eta(T_{\max}, \vec{u})) \geq d_k$. Since $\eta(t, \vec{u}) \in \mathcal{N}_b$ for any $t \in [0, T_{\max})$, we deduce from Lemma 2.11 that

$$
\begin{aligned}
J_\beta\left(\eta\left(T_{\max}, \vec{u}\right)\right) &= J_\beta(\vec{u}) - \int_0^{T_{\max}} J_\beta'(\eta(t, \vec{u}))[V(\eta(t, \vec{u}))] \, dt \\
&\leq J_\beta(\vec{u}) - \frac{C_0}{2} \int_0^{T_{\max}} \|V(\eta(t, \vec{u}))\|_H^2 \, dt \leq J_\beta(\vec{u}) < d_k,
\end{aligned} \tag{2.28}
$$

a contradiction. So $T_{\max} = +\infty$. Then similarly as (2.28) we have $J_\beta(\eta(t, \vec{u})) \leq J_\beta(\vec{u}) < d_k$ for all $t > 0$, so $\eta(t, \vec{u}) \in \mathcal{N}_{b,\beta}$ and then $(i), (iii)$ hold.

By (2.22) we have $V(\sigma_i(\vec{u})) = \sigma_i(V(\vec{u}))$. Then by the uniqueness of solutions of the initial value problem (2.27), it is easy to check that (ii) holds.

Finally, let $\delta_0 \in (0, 2^{-1/4})$ such that Lemma 2.9 holds for every $\delta < \delta_0$. For any $\vec{u} \in \mathcal{N}_{b,\beta}$ with $\mathrm{dist}_4(\vec{u}, \mathcal{P}) = \delta < \delta_0$, since

$$\eta(t, \vec{u}) = \vec{u} + t\frac{d}{dt}\eta(0, \vec{u}) + o(t) = \vec{u} - tV(\vec{u}) + o(t) = (1-t)\vec{u} + tK(\vec{u}) + o(t),$$

so we see from Lemma 2.9 that

$$\begin{aligned}\mathrm{dist}_4(\eta(t, \vec{u}), \mathcal{P}) &= \mathrm{dist}_4((1-t)\vec{u} + tK(\vec{u}) + o(t), \mathcal{P}) \\ &\le (1-t)\mathrm{dist}_4(\vec{u}, \mathcal{P}) + t\mathrm{dist}_4(K(\vec{u}), \mathcal{P}) + o(t) \\ &\le (1-t)\delta + t\delta/2 + o(t) < \delta\end{aligned}$$

for $t > 0$ sufficiently small. Hence (iv) holds. □

Proof of Theorem 1.1.

Step 1. Fix any $k_1, k_2 \in [2, k+1]$ and take any $\delta \in (0, \delta_0)$. We prove that (1.2) has a sign-changing solution $(\tilde{u}_1, \tilde{u}_2) \in H$ such that $E_\beta(\tilde{u}_1, \tilde{u}_2) = c_{\beta,\delta}^{k_1,k_2}$.

Write $c_{\beta,\delta}^{k_1,k_2}$ simply by c in this step. Recall that $c < d_k$. We claim that there exists a sequence $\{\vec{u}_n : n \ge 1\} \subset \mathcal{N}_{b,\beta}$ such that

$$J_\beta(\vec{u}_n) \to c,\ V(\vec{u}_n) \to 0 \text{ as } n \to \infty, \text{ and } \mathrm{dist}_4(\vec{u}_n, \mathcal{P}) \ge \delta,\ \forall\, n \in \mathbb{N}. \tag{2.29}$$

If (2.29) does not hold, there exists small $\varepsilon \in (0, 1)$ such that

$$\|V(\vec{u})\|_H^2 \ge \varepsilon, \quad \forall\, u \in \mathcal{N}_{b,\beta},\ |J_\beta(\vec{u}) - c| \le 2\varepsilon,\ \mathrm{dist}_4(\vec{u}, \mathcal{P}) \ge \delta.$$

Recalling the definition of c in (2.25), we see that there exists $A \in \Gamma_\beta^{(k_1,k_2)}$ such that

$$\sup_{A \setminus \mathcal{P}_\delta} J_\beta < c + \varepsilon.$$

Since $\sup_A J_\beta < d_k$, so $A \subset \mathcal{N}_{b,\beta}$. Then we can consider $B = \eta(4/C_0, A)$, where η is in Lemma 2.12 and C_0 is in Lemma 2.2. Lemma 2.12-(i) yields $B \subset \mathcal{N}_{b,\beta}$. By Lemma 2.6-$(ii)$ and Lemma 2.12-(ii) we have $B \in \Gamma^{(k_1,k_2)}$. Again by Lemma 2.12-(iii), we have $\sup_B J_\beta \le \sup_A J_\beta < d_k$, that is $B \in \Gamma_\beta^{(k_1,k_2)}$ and so $\sup_{B \setminus \mathcal{P}_\delta} J_\beta \ge c$. Then by Lemma 2.7 we can take $\vec{u} \in A$ such that $\eta(4/C_0, \vec{u}) \in B \setminus \mathcal{P}_\delta$ and

$$c - \varepsilon \le \sup_{B \setminus \mathcal{P}_\delta} J_\beta - \varepsilon < J_\beta(\eta(4/C_0, \vec{u})).$$

Since $\eta(t, \vec{u}) \in \mathcal{N}_{b,\beta}$ for any $t \ge 0$, Lemma 2.12-(iv) yields $\eta(t, \vec{u}) \notin \mathcal{P}_\delta$ for any $t \in [0, 4/C_0]$. In particular, $\vec{u} \notin \mathcal{P}_\delta$ and so $J_\beta(\vec{u}) < c + \varepsilon$. Then for any $t \in [0, 4/C_0]$, we have

$$c - \varepsilon < J_\beta(\eta(4/C_0, \vec{u})) \le J_\beta(\eta(t, \vec{u})) \le J_\beta(\vec{u}) < c + \varepsilon,$$

which implies $\|V(\eta(t,\vec{u}))\|_H^2 \geq \varepsilon$ and

$$\frac{d}{dt}J_\beta(\eta(t,\vec{u})) = -J'_\beta(\eta(t,\vec{u}))[V(\eta(t,\vec{u}))] \leq -\frac{C_0}{2}\|V(\eta(t,\vec{u}))\|_H^2 \leq -\frac{C_0}{2}\varepsilon$$

for every $t \in [0, 4/C_0]$. Hence,

$$c - \varepsilon < J_\beta(\eta(4/C_0,\vec{u})) \leq J_\beta(\vec{u}) - \int_0^{4/C_0} \frac{C_0}{2}\varepsilon\, dt < c + \varepsilon - 2\varepsilon = c - \varepsilon,$$

a contradiction. Therefore (2.29) holds. By Lemma 2.10, up to a subsequence, there exists $\vec{u} = (u_1, u_2) \in \mathcal{N}_{b,\beta}$ such that $\vec{u}_n \to \vec{u}$ strongly in H and $V(\vec{u}) = 0$, $J_\beta(\vec{u}) = c = c_{\beta,\delta}^{k_1,k_2}$. Since $\mathrm{dist}_4(\vec{u}_n, \mathcal{P}) \geq \delta$, so $\mathrm{dist}_4(\vec{u}, \mathcal{P}) \geq \delta$, which implies that both u_1 and u_2 are sign-changing. Since $V(\vec{u}) = 0$, so $\vec{u} = K(\vec{u})$. Combining this with (2.19)-(2.20), we see that $\vec{u}$ satisfies

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \alpha_1\mu_1 t_1(\vec{u})u_1^3 + \beta t_2(\vec{u})u_2^2 u_1, \\ -\Delta u_2 + \lambda_2 u_2 = \alpha_2\mu_2 t_2(\vec{u})u_2^3 + \beta t_1(\vec{u})u_1^2 u_2. \end{cases} \tag{2.30}$$

Recall that $|u_i|_4 = 1$ and $t_i(\vec{u})$ satisfies (2.4). Multiplying (2.30) by u_i and integrating over Ω, we obtain that $\alpha_1 = \alpha_2 = 1$. Again by (2.30), we see that $(\tilde{u}_1, \tilde{u}_2) := (\sqrt{t_1(\vec{u})}u_1, \sqrt{t_2(\vec{u})}u_2)$ is a sign-changing solution of the original problem (1.2). Moreover, (2.5) and (2.10) yield $E_\beta(\tilde{u}_1, \tilde{u}_2) = J_\beta(u_1, u_2) = c_{\beta,\delta}^{k_1,k_2}$.

Step 2. We prove that (1.2) has at least k sign-changing solutions.

Assume by contradiction that (1.2) has at most $k-1$ sign-changing solutions. Fix any $k_2 \in [2, k+1]$ and $\delta \in (0, \delta_0)$. Since $\Gamma_\beta^{(k_1+1,k_2)} \subset \Gamma_\beta^{(k_1,k_2)}$, we have

$$c_{\beta,\delta}^{2,k_2} \leq c_{\beta,\delta}^{3,k_2} \leq \cdots \leq c_{\beta,\delta}^{k,k_2} \leq c_{\beta,\delta}^{k+1,k_2} < d_k. \tag{2.31}$$

Since $c_{\beta,\delta}^{k_1,k_2}$ is a sign-changing critical value of E_β for each $k_1 \in [2, k+1]$ (that is, E_β has a sign-changing critical point $\vec{u}$ with $E_\beta(\vec{u}) = c_{\beta,\delta}^{k_1,k_2}$), by (2.31) and our assumption that (1.2) has at most $k-1$ sign-changing solutions, there exists some $2 \leq N_1 \leq k$ such that

$$c_{\beta,\delta}^{N_1,k_2} = c_{\beta,\delta}^{N_1+1,k_2} =: \bar{c} < d_k. \tag{2.32}$$

Define

$$\mathcal{K} := \{\vec{u} \in \mathcal{N}_b \;:\; \vec{u} \text{ sign-changing},\;\; J_\beta(\vec{u}) = \bar{c},\;\; V(\vec{u}) = 0\}. \tag{2.33}$$

Then $\mathcal{K}$ is finite. By (2.22) one has that $\sigma_i(\vec{u}) \in \mathcal{K}$ if $\vec{u} \in \mathcal{K}$, that is, $\mathcal{K} \subset \mathcal{F}$. Hence there exist $k_0 \leq k-1$ and $\{\vec{u}_m : 1 \leq m \leq k_0\} \subset \mathcal{K}$ such that

$$\mathcal{K} = \{\vec{u}_m,\, \sigma_1(\vec{u}_m),\, \sigma_2(\vec{u}_m),\, -\vec{u}_m \;:\; 1 \leq m \leq k_0\}.$$

Then there exist open neighborhoods $O_{\vec{u}_m}$ of $\vec{u}_m$ in H, such that any two of $\overline{O_{\vec{u}_m}}$, $\sigma_1(\overline{O_{\vec{u}_m}})$, $\sigma_2(\overline{O_{\vec{u}_m}})$ and $-\overline{O_{\vec{u}_m}}$, where $1 \leq m \leq k_0$, are disjointed and

$$\mathcal{K} \subset O := \bigcup_{m=1}^{k_0} O_{\vec{u}_m} \cup \sigma_1(O_{\vec{u}_m}) \cup \sigma_2(O_{\vec{u}_m}) \cup -O_{\vec{u}_m}.$$

Define a continuous map $\tilde{f}:\overline{O}\to\mathbb{R}\setminus\{0\}$ by

$$\tilde{f}(\vec{u}):=\begin{cases}1, & \text{if } \vec{u}\in\bigcup_{m=1}^{k_0}\overline{O_{\vec{u}_m}}\cup\sigma_2(\overline{O_{\vec{u}_m}}),\\ -1, & \text{if } \vec{u}\in\bigcup_{m=1}^{k_0}\sigma_1(\overline{O_{\vec{u}_m}})\cup-\overline{O_{\vec{u}_m}}.\end{cases}$$

Then $\tilde{f}(\sigma_1(\vec{u}))=-\tilde{f}(\vec{u})$ and $\tilde{f}(\sigma_2(\vec{u}))=\tilde{f}(\vec{u})$. By Tietze's extension theorem, there exists $f\in C(H,\mathbb{R})$ such that $f|_O\equiv\tilde{f}$. Define

$$F(\vec{u}):=\frac{f(\vec{u})+f(\sigma_2(\vec{u}))-f(\sigma_1(\vec{u}))-f(-\vec{u})}{4},$$

then $F|_O\equiv\tilde{f}$, $F(\sigma_1(\vec{u}))=-F(\vec{u})$ and $F(\sigma_2(\vec{u}))=F(\vec{u})$. Define

$$\mathcal{K}_\tau:=\left\{\vec{u}\in\mathcal{N}_b:\inf_{\vec{v}\in\mathcal{K}}\|\vec{u}-\vec{v}\|_H<\tau\right\}.$$

Then we can take small $\tau>0$ such that $\mathcal{K}_{2\tau}\subset O$. Recalling $V(\vec{u})=0$ in $\mathcal{K}$ and $\mathcal{K}$ finite, we see that there exists $\widetilde{C}>0$ such that

$$\|V(\vec{u})\|_H\le\widetilde{C},\quad\forall\,\vec{u}\in\overline{\mathcal{K}_{2\tau}}.\tag{2.34}$$

For any $\vec{u}\in\mathcal{K}_{2\tau}$, we have $F(\vec{u})=\tilde{f}(\vec{u})\neq0$. That is $F(\mathcal{K}_{2\tau})\subset\mathbb{R}\setminus\{0\}$. By (2.33) and Lemma 2.10 there exists small $\varepsilon\in(0,(d_k-\bar{c})/2)$ such that

$$\|V(\vec{u})\|_H^2\ge\varepsilon,\quad\forall\,u\in\mathcal{N}_b\setminus(\mathcal{K}_\tau\cup\mathcal{P}_\delta)\ \text{satisfying}\ |J_\beta(\vec{u})-\bar{c}|\le2\varepsilon.\tag{2.35}$$

Recalling C_0 in Lemma 2.2, we let

$$\alpha:=\frac{1}{2}\min\left\{1,\frac{\tau C_0}{2\widetilde{C}}\right\}.\tag{2.36}$$

By (2.25)-(2.26) and (2.32) we take $A\in\Gamma_\beta^{(N_1+1,k_2)}$ such that

$$\sup_{A\setminus\mathcal{P}_\delta}J_\beta<c_{\beta,\delta}^{N_1+1,k_2}+\alpha\varepsilon/2=\bar{c}+\alpha\varepsilon/2.\tag{2.37}$$

Let $B:=A\setminus\mathcal{K}_{2\tau}$, then it is easy to check that $B\subset\mathcal{F}$. We claim that $\vec{\gamma}(B)\ge(N_1,k_2)$. If not, there exists $\tilde{g}\in F_{(N_1,k_2)}(B)$ such that $\tilde{g}(\vec{u})\neq0$ for any $\vec{u}\in B$. By Tietze's extension theorem, there exists $\bar{g}=(\bar{g}_1,\bar{g}_2)\in C(H,\mathbb{R}^{N_1-1}\times\mathbb{R}^{k_2-1})$ such that $\bar{g}|_B\equiv\tilde{g}$. Define $g=(g_1,g_2)\in C(H,\mathbb{R}^{N_1-1}\times\mathbb{R}^{k_2-1})$ by

$$g_1(\vec{u}):=\frac{\bar{g}_1(\vec{u})+\bar{g}_1(\sigma_2(\vec{u}))-\bar{g}_1(\sigma_1(\vec{u}))-\bar{g}_1(-\vec{u})}{4},$$
$$g_2(\vec{u}):=\frac{\bar{g}_2(\vec{u})+\bar{g}_2(\sigma_1(\vec{u}))-\bar{g}_2(\sigma_2(\vec{u}))-\bar{g}_2(-\vec{u})}{4},$$

then $g|_B\equiv\tilde{g}$, $g_i(\sigma_i(\vec{u}))=-g_i(\vec{u})$ and $g_i(\sigma_j(\vec{u}))=g_i(\vec{u})$ for $j\neq i$. Finally we define $G=(G_1,G_2)\in C(A,\,\mathbb{R}^{N_1+1-1}\times\mathbb{R}^{k_2-1})$ by

$$G_1(\vec{u}):=(F(\vec{u}),\,g_1(\vec{u}))\in\mathbb{R}^{N_1+1-1},\quad G_2(\vec{u}):=g_2(\vec{u})\in\mathbb{R}^{k_2-1}.$$

By our constructions of F and g, we have $G \in F_{(N_1+1,k_2)}(A)$. Since $\vec{\gamma}(A) \geq (N_1+1,k_2)$, so $G(\vec{u}) = 0$ for some $\vec{u} \in A$. If $\vec{u} \in \mathcal{K}_{2\tau}$, then $F(\vec{u}) \neq 0$, a contradiction. So $\vec{u} \in A \setminus \mathcal{K}_{2\tau} = B$, and then $g(\vec{u}) = \tilde{g}(\vec{u}) \neq 0$, also a contradiction. Hence $\vec{\gamma}(B) \geq (N_1,k_2)$. Note that $B \subset A \subset \mathcal{N}_b$ and $\sup_B J_\beta \leq \sup_A J_\beta < d_k$, we see that $B \subset \mathcal{N}_{b,\beta}$ and $B \in \Gamma_\beta^{(N_1,k_2)}$. Then we can consider $D := \eta(\tau/(2\widetilde{C}), B)$, where η is in Lemma 2.12 and $\widetilde{C}$ is in (2.34). By Lemma 2.6-(ii) and Lemma 2.12 we have $D \subset \mathcal{N}_{b,\beta}$, $D \in \Gamma^{(N_1,k_2)}$ and $\sup_D J_\beta \leq \sup_B J_\beta < d_k$, that is $D \in \Gamma_\beta^{(N_1,k_2)}$. Then we see from (2.25)-(2.26) and (2.32) that

$$\sup_{D\setminus\mathcal{P}_\delta} J_\beta \geq c_{\beta,\delta}^{N_1,k_2} = \bar{c}.$$

By Lemma 2.7 we can take $\vec{u} \in B$ such that $\eta(\tau/(2\widetilde{C}), \vec{u}) \in D \setminus \mathcal{P}_\delta$ and

$$\bar{c} - \alpha\varepsilon/2 \leq \sup_{D\setminus\mathcal{P}_\delta} J_\beta - \alpha\varepsilon/2 < J_\beta(\eta(\tau/(2\widetilde{C}), \vec{u})).$$

Since $\eta(t,\vec{u}) \in \mathcal{N}_{b,\beta}$ for any $t \geq 0$, Lemma 2.12-(iv) yields $\eta(t,\vec{u}) \notin \mathcal{P}_\delta$ for any $t \in [0, \tau/(2\widetilde{C})]$. In particular, $\vec{u} \notin \mathcal{P}_\delta$ and so (2.37) yields $J_\beta(\vec{u}) < \bar{c} + \alpha\varepsilon/2$. Then for any $t \in [0, \tau/(2\widetilde{C})]$, we have

$$\bar{c} - \alpha\varepsilon/2 < J_\beta(\eta(\tau/(2\widetilde{C}), \vec{u})) \leq J_\beta(\eta(t,\vec{u})) \leq J_\beta(\vec{u}) < \bar{c} + \alpha\varepsilon/2.$$

Recall that $\vec{u} \in B = A \setminus \mathcal{K}_{2\tau}$. If there exists $T \in (0, \tau/(2\widetilde{C}))$ such that $\eta(T,\vec{u}) \in \mathcal{K}_\tau$, then there exist $0 \leq t_1 < t_2 \leq T$ such that $\eta(t_1,\vec{u}) \in \partial\mathcal{K}_{2\tau}$, $\eta(t_2,\vec{u}) \in \partial\mathcal{K}_\tau$ and $\eta(t,\vec{u}) \in \mathcal{K}_{2\tau} \setminus \mathcal{K}_\tau$ for any $t \in (t_1,t_2)$. So we see from (2.34) that

$$\tau \leq \|\eta(t_1,\vec{u}) - \eta(t_2,\vec{u})\|_H = \left\| \int_{t_1}^{t_2} V(\eta(t,\vec{u}))\, dt \right\|_H \leq 2\widetilde{C}(t_2 - t_1),$$

that is, $\tau/(2\widetilde{C}) \leq t_2 - t_1 \leq T$, a contradiction. Hence $\eta(t,\vec{u}) \notin \mathcal{K}_\tau$ for any $t \in (0, \tau/(2\widetilde{C}))$. Then as Step 1, we deduce from (2.35) and (2.36) that

$$\bar{c} - \frac{\alpha\varepsilon}{2} < J_\beta(\eta(\tau/(2\widetilde{C}), \vec{u})) \leq J_\beta(\vec{u}) - \int_0^{\frac{\tau}{2\widetilde{C}}} \frac{C_0}{2}\varepsilon\, dt < \bar{c} + \frac{\alpha\varepsilon}{2} - \alpha\varepsilon = \bar{c} - \frac{\alpha\varepsilon}{2},$$

a contradiction. Hence (1.2) has at least k sign-changing solutions for any $\beta \in (0, \beta_k)$. This completes the proof. □

3 Proof of Theorem 1.2

Let $k = 1$ in Section 2. By the proof of Theorem 1.1 there exists $\beta_1 > 0$ such that, for any $\beta \in (0, \beta_1)$, (1.2) has a sign-changing solution $(u_{\beta,1}, v_{\beta,1})$ with $E_\beta(u_{\beta,1}, v_{\beta,1}) = c_{\beta,\delta}^{2,2} < d_1$. Recalling S in (2.14), we define

$$\beta_1' := \min\left\{S^2/(4d_1),\, \beta_1\right\}. \tag{3.1}$$

Fix any $\beta \in (0, \beta_1')$ and define

$$c_\beta := \inf_{\vec{u} \in \mathcal{K}_\beta} E_\beta(\vec{u}); \quad \mathcal{K}_\beta := \{\vec{u} : \vec{u} \text{ is a sign-changing solution of (1.2)}\}.$$

Then $\mathcal{K}_\beta \neq \emptyset$ and $c_\beta < d_1$. Let $\vec{u}_n = (u_{n,1}, u_{n,2}) \in \mathcal{K}_\beta$ be a minimizing sequence of c_β with $E_\beta(\vec{u}_n) < d_1$ for all $n \geq 1$. Then $\|u_{n,1}\|_{\lambda_1}^2 + \|u_{n,2}\|_{\lambda_2}^2 < 4d_1$. Up to a subsequence, we may assume that $\vec{u}_n \to \vec{u} = (u_1, u_2)$ weakly in H and strongly in $L^4(\Omega) \times L^4(\Omega)$. Since $E_\beta'(\vec{u}_n) = 0$, it is standard to prove that $\vec{u}_n \to \vec{u} = (u_1, u_2)$ strongly in H, $E_\beta'(\vec{u}) = 0$ and $E_\beta(\vec{u}) = c_\beta$. On the other hand, we deduce from $E_\beta'(\vec{u}_n)(u_{n,1}^\pm, 0) = 0$ and $E_\beta'(\vec{u}_n)(0, u_{n,2}^\pm) = 0$ that

$$\begin{aligned} S|u_{n,i}^\pm|_4^2 \leq \|u_{n,i}^\pm\|_{\lambda_i}^2 &= \mu_i |u_{n,i}^\pm|_4^4 + \beta \int_\Omega |u_{n,i}^\pm|^2 u_{n,j}^2 dx \leq \mu_i |u_{n,i}^\pm|_4^4 + \beta |u_{n,i}^\pm|_4^2 |u_{n,j}|_4^2 \\ &\leq \mu_i |u_{n,i}^\pm|_4^4 + \frac{\beta}{S} |u_{n,i}^\pm|_4^2 \|u_{n,j}\|_{\lambda_j}^2 < \mu_i |u_{n,i}^\pm|_4^4 + \frac{4d_1\beta}{S} |u_{n,i}^\pm|_4^2, \end{aligned}$$

which implies that $|u_{n,i}^\pm|_4 \geq C > 0$ for all $n \geq 1$ and $i = 1, 2$, where C is a constant independent of n and i. Hence $|u_i^\pm|_4 \geq C$ and so $\vec{u}$ is a least energy sign-changing solution of (1.2). $\square$

4 Proof of Theorems 1.3

The following arguments are similar to those in Section 2 with some modifications. Here, although some definitions are slight different from those in Section 2, we will use the same notations as in Section 2 for convenience. To obtain semi-nodal solutions (u_1, u_2) such that u_1 changes sign and u_2 is positive, we consider the following functional

$$\widetilde{E}_\beta(u_1, u_2) := \frac{1}{2}\left(\|u_1\|_{\lambda_1}^2 + \|u_2\|_{\lambda_2}^2\right) - \frac{1}{4}\left(\mu_1 |u_1|_4^4 + \mu_2 |u_2^+|_4^4\right) - \frac{\beta}{2}\int_\Omega u_1^2 u_2^2 \, dx,$$

and modify the definition of $\widetilde{H}$ by $\widetilde{H} := \{(u_1, u_2) \in H : u_1 \neq 0, \ u_2^+ \neq 0\}$. Then by similar proofs as in Section 2, we have the following lemma.

Lemma 4.1. *For any $\vec{u} = (u_1, u_2) \in \widetilde{H}$, if*

$$\begin{cases} \mu_2 |u_2^+|_4^4 \|u_1\|_{\lambda_1}^2 - \beta \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2 \, dx > 0, \\ \mu_1 |u_1|_4^4 \|u_2\|_{\lambda_2}^2 - \beta \|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2 u_2^2 \, dx > 0, \end{cases} \tag{4.1}$$

then system

$$\begin{cases} \|u_1\|_{\lambda_1}^2 = t_1 \mu_1 |u_1|_4^4 + t_2 \beta \int_\Omega u_1^2 u_2^2 \, dx \\ \|u_2\|_{\lambda_2}^2 = t_2 \mu_2 |u_2^+|_4^4 + t_1 \beta \int_\Omega u_1^2 u_2^2 \, dx \end{cases} \tag{4.2}$$

has a unique solution

$$\begin{cases} t_1(\vec{u}) = \frac{\mu_2 |u_2^+|_4^4 \|u_1\|_{\lambda_1}^2 - \beta \|u_2\|_{\lambda_2}^2 \int_\Omega u_1^2 u_2^2 \, dx}{\mu_1 \mu_2 |u_1|_4^4 |u_2^+|_4^4 - \beta^2 (\int_\Omega u_1^2 u_2^2 \, dx)^2} > 0 \\ t_2(\vec{u}) = \frac{\mu_1 |u_1|_4^4 \|u_2\|_{\lambda_2}^2 - \beta \|u_1\|_{\lambda_1}^2 \int_\Omega u_1^2 u_2^2 \, dx}{\mu_1 \mu_2 |u_1|_4^4 |u_2^+|_4^4 - \beta^2 (\int_\Omega u_1^2 u_2^2)^2 \, dx} > 0. \end{cases} \tag{4.3}$$

Moreover,

$$\sup_{t_1,t_2\geq 0} \widetilde{E}_\beta\left(\sqrt{t_1}u_1,\,\sqrt{t_2}u_2\right) = \widetilde{E}_\beta\left(\sqrt{t_1(\vec{u})}u_1,\sqrt{t_2(\vec{u})}u_2\right)$$
$$= \frac{1}{4}\frac{\mu_2|u_2^+|_4^4\|u_1\|_{\lambda_1}^4 - 2\beta\|u_1\|_{\lambda_1}^2\|u_2\|_{\lambda_2}^2\int_\Omega u_1^2u_2^2\,dx + \mu_1|u_1|_4^4\|u_2\|_{\lambda_2}^4}{\mu_1\mu_2|u_1|_4^4|u_2^+|_4^4 - \beta^2(\int_\Omega u_1^2u_2^2\,dx)^2} \quad (4.4)$$

and $(t_1(\vec{u}), t_2(\vec{u}))$ *is the unique maximum point of* $\widetilde{E}_\beta(\sqrt{t_1}u_1, \sqrt{t_2}u_2)$*.*

Now we modify the definitions of $\mathcal{M}^*$, $\mathcal{M}^*_\beta$, $\mathcal{M}^{**}_\beta$, $\mathcal{M}$ and $\mathcal{M}_\beta$ by

$$\mathcal{M}^* := \left\{\vec{u}\in H \;:\; 1/2<|u_1|_4^4<2,\ 1/2<|u_2^+|_4^4<2\right\}; \quad (4.5)$$
$$\mathcal{M}^*_\beta := \{\vec{u}\in\mathcal{M}^* \;:\; \vec{u} \text{ satisfies } (4.1)\};$$
$$\mathcal{M}^{**}_\beta := \left\{\vec{u}\in\mathcal{M}^* \;:\; \begin{matrix}\mu_2\|u_1\|_{\lambda_1}^2 - \beta\|u_2\|_{\lambda_2}^2\int_\Omega u_1^2u_2^2\,dx>0\\ \mu_1\|u_2\|_{\lambda_2}^2 - \beta\|u_1\|_{\lambda_1}^2\int_\Omega u_1^2u_2^2\,dx>0\end{matrix}\right\};$$
$$\mathcal{M} := \left\{\vec{u}\in H \;:\; |u_1|_4=1,\ |u_2^+|_4=1\right\}, \quad \mathcal{M}_\beta := \mathcal{M}\cap\mathcal{M}^*_\beta, \quad (4.6)$$

and define a new functional $J_\beta : \mathcal{M}^{**}_\beta \to (0,+\infty)$ as in Section 2 by

$$J_\beta(\vec{u}) := \frac{1}{4}\frac{\mu_2\|u_1\|_{\lambda_1}^4 - 2\beta\|u_1\|_{\lambda_1}^2\|u_2\|_{\lambda_2}^2\int_\Omega u_1^2u_2^2\,dx + \mu_1\|u_2\|_{\lambda_2}^4}{\mu_1\mu_2 - \beta^2(\int_\Omega u_1^2u_2^2\,dx)^2}.$$

Then $J_\beta \in C^1(\mathcal{M}^{**}_\beta,\,(0,+\infty))$ and (2.8)-(2.9) hold for any $\vec{u}\in\mathcal{M}_\beta$ and $\varphi,\,\psi\in H_0^1(\Omega)$. Moreover, Lemma 4.1 yields

$$J_\beta(u_1,u_2) = \sup_{t_1,t_2\geq 0}\widetilde{E}_\beta\left(\sqrt{t_1}u_1,\,\sqrt{t_2}u_2\right), \quad \forall\,(u_1,u_2)\in\mathcal{M}_\beta. \quad (4.7)$$

Under this new definitions (4.5)-(4.6), we define $\mathcal{N}_b^*$ and $\mathcal{N}_b$ as in (2.11)-(2.13). Since $|u_2|_4^2 \leq S^{-1}\|u_2\|_{\lambda_2}^2 \leq b/S$ for all $\vec{u}\in\mathcal{N}_b^*$, by trivial modifications it is easy to check that Lemmas 2.2 and 2.3 also hold here. Moreover, we may assume that (2.16) also holds here for any $\beta\in(0,\beta_k)$.

Now we fix any $\beta\in(0,\beta_k)$. For any $\vec{u}=(u_1,u_2)\in\mathcal{N}_b^*$, let $\tilde{w}_i\in H_0^1(\Omega)$, $i=1,2$, be the unique solutions of the following linear problem

$$\begin{cases}-\Delta\tilde{w}_1+\lambda_1\tilde{w}_1-\beta t_2(\vec{u})u_2^2\tilde{w}_1=\mu_1t_1(\vec{u})u_1^3, & \tilde{w}_1\in H_0^1(\Omega),\\ -\Delta\tilde{w}_2+\lambda_2\tilde{w}_2-\beta t_1(\vec{u})u_1^2\tilde{w}_2=\mu_2t_2(\vec{u})(u_2^+)^3, & \tilde{w}_2\in H_0^1(\Omega).\end{cases} \quad (4.8)$$

As in Section 2, we define

$$w_i=\alpha_i\tilde{w}_i, \quad \text{where } \alpha_1=\frac{1}{\int_\Omega u_1^3\tilde{w}_1\,dx}>0,\ \alpha_2=\frac{1}{\int_\Omega(u_2^+)^3\tilde{w}_2\,dx}>0. \quad (4.9)$$

Then (w_1,w_2) is the unique solution of the problem

$$\begin{cases}-\Delta w_1+\lambda_1w_1-\beta t_2(\vec{u})u_2^2w_1=\alpha_1\mu_1t_1(\vec{u})u_1^3, & w_1\in H_0^1(\Omega),\\ -\Delta w_2+\lambda_2w_2-\beta t_1(\vec{u})u_1^2w_2=\alpha_2\mu_2t_2(\vec{u})(u_2^+)^3, & w_2\in H_0^1(\Omega),\\ \int_\Omega u_1^3w_1\,dx=1, & \int_\Omega(u_2^+)^3w_2\,dx=1.\end{cases} \quad (4.10)$$

As in Section 2, the operator $K = (K_1, K_2) : \mathcal{N}_b^* \to H$ is defined as $K(\vec{u}) := \vec{w} = (w_1, w_2)$, and similar arguments as Lemma 2.4 yield $K \in C^1(\mathcal{N}_b^*, H)$. Since $u_n \to u$ in $L^4(\Omega)$ implies $u_n^+ \to u^+$ in $L^4(\Omega)$, so Lemma 2.5 also holds for this new K defined here. Clearly

$$K(\sigma_1(\vec{u})) = \sigma_1(K(\vec{u})). \tag{4.11}$$

Remark that (4.11) only holds for σ_1 and in the sequel we only use σ_1. Consider

$$\mathcal{F} = \{A \subset \mathcal{M} : A \text{ is closed and } \sigma_1(\vec{u}) \in A \ \forall \, \vec{u} \in A\},$$

and, for each $A \in \mathcal{F}$ and $k_1 \geq 2$, the class of functions

$$F_{(k_1,1)}(A) = \left\{ f : A \to \mathbb{R}^{k_1 - 1} \ : \ f \text{ continuous and } f(\sigma_1(\vec{u})) = -f(\vec{u}) \right\}.$$

Definition 4.1. *(Modified vector genus, slightly different from Definition 2.1) Let $A \in \mathcal{F}$ and take any $k_1 \in \mathbb{N}$ with $k_1 \geq 2$. We say that $\vec{\gamma}(A) \geq (k_1, 1)$ if for every $f \in F_{(k_1,1)}(A)$ there exists $\vec{u} \in A$ such that $f(\vec{u}) = 0$. We denote*

$$\Gamma^{(k_1,1)} := \{A \in \mathcal{F} : \vec{\gamma}(A) \geq (k_1, 1)\}.$$

Lemma 4.2. *(see [10, Lemma 4.2]) With the previous notations, the following properties hold.*

(i) Take $A := A_1 \times A_2 \subset \mathcal{M}$ and let $\eta : S^{k_1 - 1} \to A_1$ be a homeomorphism such that $\eta(-x) = -\eta(x)$ for every $x \in S^{k_1 - 1}$. Then $A \in \Gamma^{(k_1,1)}$.

(ii) We have $\overline{\eta(A)} \in \Gamma^{(k_1,1)}$ whenever $A \in \Gamma^{(k_1,1)}$ and a continuous map $\eta : A \to \mathcal{M}$ is such that $\eta \circ \sigma_1 = \sigma_1 \circ \eta$.

Now we modify the definitions of $\mathcal{P}$ and $\text{dist}_4(\vec{u}, \mathcal{P})$ in (2.23)-(2.24) by

$$\mathcal{P} := \mathcal{P}_1 \cup -\mathcal{P}_1, \quad \text{dist}_4(\vec{u}, \mathcal{P}) := \min\left\{\text{dist}_4(u_1, \mathcal{P}_1), \ \text{dist}_4(u_1, -\mathcal{P}_1)\right\}. \tag{4.12}$$

Under this new definition, u_1 changes sign if $\text{dist}_4(\vec{u}, \mathcal{P}) > 0$.

Lemma 4.3. *(see [10, Lemma 4.3]) Let $k_1 \geq 2$. Then for any $\delta < 2^{-1/4}$ and any $A \in \Gamma^{(k_1,1)}$ there holds $A \setminus \mathcal{P}_\delta \neq \emptyset$.*

Lemma 4.4. *There exists $A \in \Gamma^{(k+1,1)}$ such that $A \subset \mathcal{N}_b$ and $\sup_A J_\beta < d_k$.*

Proof. Recalling $\varphi_0 \in W_{k+1}$ is positive, we define

$$A_1 := \left\{u \in W_{k+1} \ : \ |u|_4 = 1\right\}, \quad A_2 := \{C\varphi_0 : C = 1/|\varphi_0|_4\}.$$

Then by Lemma 4.2-(i) one has $A := A_1 \times A_2 \in \Gamma^{(k+1,1)}$. The rest of the proof is the same as Lemma 2.8. $\square$

For every $k_1 \in [2, k+1]$ and $0 < \delta < 2^{-1/4}$, we define

$$c_{\beta,\delta}^{k_1,1} := \inf_{A \in \Gamma_\beta^{(k_1,1)}} \sup_{\vec{u} \in A \setminus \mathcal{P}_\delta} J_\beta(\vec{u}),$$

where the definition of $\Gamma_\beta^{(k_1,1)}$ is the same as (2.26). Then Lemma 4.4 yields $\Gamma_\beta^{(k_1,1)} \neq \emptyset$ and so $c_{\beta,\delta}^{k_1,1}$ is well defined for each $k_1 \in [2, k+1]$. Moreover, $c_{\beta,\delta}^{k_1,1} < d_k$ for any $\delta \in (0, 2^{-1/4})$ and $k_1 \in [2, k+1]$. Define $\mathcal{N}_{b,\beta} := \{\vec{u} \in \mathcal{N}_b : J_\beta(\vec{u}) < d_k\}$ as in Section 2. Under the new definition (4.12), it is easy to see that Lemma 2.9 also holds here. Now as in Section 2, we define a map $V : \mathcal{N}_b^* \to H$ by $V(\vec{u}) := \vec{u} - K(\vec{u})$. Then Lemma 2.10 also holds here. Recall from (4.6) and (4.10) that $\int_\Omega (u_2^+)^3(u_2 - w_2)\,dx = 1 - 1 = 0$ for any $\vec{u} = (u_1, u_2) \in \mathcal{N}_b$. Then by similar arguments, we see that Lemma 2.11 also holds here.

Lemma 4.5. *There exists a unique global solution $\eta = (\eta_1, \eta_2) : [0,\infty)\times\mathcal{N}_{b,\beta} \to H$ for the initial value problem*

$$\frac{d}{dt}\eta(t,\vec{u}) = -V(\eta(t,\vec{u})), \quad \eta(0,\vec{u}) = \vec{u} \in \mathcal{N}_{b,\beta}. \tag{4.13}$$

Moreover, conclusions (i), (iii) and (iv) of Lemma 2.12 also hold here, and $\eta(t, \sigma_1(\vec{u})) = \sigma_1(\eta(t,\vec{u}))$ for any $t > 0$ and $u \in \mathcal{N}_{b,\beta}$.

Proof. Recalling $V(\vec{u}) \in C^1(\mathcal{N}_b^*, H)$, we see that (4.13) has a unique solution $\eta : [0, T_{\max}) \times \mathcal{N}_{b,\beta} \to H$, where $T_{\max} > 0$ is the maximal time such that $\eta(t,\vec{u}) \in \mathcal{N}_b^*$ for all $t \in [0, T_{\max})$. Fix any $\vec{u} = (u_1, u_2) \in \mathcal{N}_{b,\beta}$, we deduce from (4.13) that $\frac{d}{dt}\int_\Omega \left(\eta_2(t,\vec{u})^+\right)^4 dx = 4 - 4\int_\Omega \left(\eta_2(t,\vec{u})^+\right)^4 dx$, $\forall\, 0 < t < T_{\max}$. Since $\int_\Omega \left(\eta_2(0,\vec{u})^+\right)^4 dx = \int_\Omega (u_2^+)^4 dx = 1$, so $\int_\Omega \left(\eta_2(t,\vec{u})^+\right)^4 dx \equiv 1$ for all $0 \le t < T_{\max}$. Recalling (4.11), we see that the rest of the proof is similar to Lemma 2.12. $\square$

Proof of Theorem 1.3. First we fix any $k_1 \in [2, k+1]$. Then by similar arguments as Step 1 in the proof of Theorem 1.1, for small $\delta > 0$, there exists $\vec{u} = (u_1, u_2) \in \mathcal{N}_b$ such that

$$J_\beta(\vec{u}) = c_{\beta,\delta}^{k_1,1}, \quad V(\vec{u}) = 0 \quad \text{and } \mathrm{dist}_4(\vec{u}, \mathcal{P}) \ge \delta.$$

So u_1 changes sign. Since $V(\vec{u}) = 0$, so $\vec{u} = K(\vec{u})$. Combining this with (4.10), we see that $\vec{u}$ satisfies

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \alpha_1\mu_1 t_1(\vec{u}) u_1^3 + \beta t_2(\vec{u}) u_2^2 u_1, \\ -\Delta u_2 + \lambda_2 u_2 = \alpha_2\mu_2 t_2(\vec{u}) (u_2^+)^3 + \beta t_1(\vec{u}) u_1^2 u_2. \end{cases} \tag{4.14}$$

Since $|u_1|_4 = 1$, $|u_2^+|_4 = 1$ and $t_i(\vec{u})$ satisfies (4.2), so $\alpha_1 = \alpha_2 = 1$. Multiplying the second equation of (4.14) by u_2^- and integrating over Ω, we see from (2.16) that $\|u_2^-\|_{\lambda_2}^2 = 0$, so $u_2 \ge 0$. By the strong maximum principle, $u_2 > 0$ in Ω. Hence $(\tilde{u}_1, \tilde{u}_2) := (\sqrt{t_1(\vec{u})}u_1, \sqrt{t_2(\vec{u})}u_2)$ is a semi-nodal solution of the original problem (1.2) with $\tilde{u}_1$ sign-changing and $\tilde{u}_2$ positive. Moreover, (4.4) and (4.7) yield $E_\beta(\tilde{u}_1, \tilde{u}_2) = \widetilde{E}_\beta(\tilde{u}_1, \tilde{u}_2) = J_\beta(u_1, u_2) = c_{\beta,\delta}^{k_1,1} < d_k$. Finally, since $k_1 \in [2, k+1]$, by similar arguments as Step 2 of proving Theorem 1.1 with trivial modifications, we can prove that (1.2) has at least k semi-nodal solutions. This completes the proof. $\square$

Remark 4.1. *By a similar argument as in Section 3, we can prove that there exists $\beta_1'' > 0$ such that for any $\beta \in (0, \beta_1'')$, (1.2) has a semi-nodal solution which has the least energy among all semi-nodal solutions.*

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